Free Vibrations of a Thin-Walled Cylindrical Shell Subjected to a Bending Moment

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The effect of a bending moment on the free vibrations of a thin-walled cylindrical shell is investigated by means of a modified Donnell equation. An experimental investigation was performed in conjunction with the analysis. The results indicate that, as the bending moment increases, some frequencies increase in magnitude while others decrease which is contrary to results obtained for symmetric loadings.

Nomenclature

a_n = coefficients in deflection function
D = flexural stiffness of cylinder wall, $[Eh^3/12(1 - \nu^2)]$, lb-in.
E = Young's modulus of cylinder material, psi
f = frequency $[f = (\omega/(2\pi)]$, cps
g = gravitational acceleration, in./sec^2
h = thickness of cylinder, in.
L = length of cylinder, in.
$m - 1$ = number of nodes in axial mode shape
M = applied bending moment, in.-lb
n = number of circumferential waves in mode shape for $M = 0$
p = external pressure, positive inward, psi
p^* = pressure parameter $\{p/E(R/h)^2[12(1 - \nu^2)]^{1/2}\}$
P = applied compressive load, lb
q = surface loading, psi
R = radius of cylinder, in.
r, s = integers
S = radius-thickness parameter $\{[12(1 - \nu^2)]^{1/2}(R/h)\}$
t = time, sec
w = radial deflection, in.
x = distance along the longitudinal axis of the cylinder, in. (see Fig. 1)
θ = angle denoting the circumferential location of a point on the cylinder middle surface radius (see Fig. 1)
$\sigma_x, \sigma_\theta, \tau_{x\theta}$ = longitudinal and circumferential direct stresses and shear stress, respectively, psi
σ_b = maximum compressive stress due to bending moment
σ_c = applied axial stress, positive for compression, $[(M/\pi R^2 h)]$, psi
$\bar{\sigma}_c$ = theoretical compressive buckling stress for a long cylinder $\{E/[3(1 - \nu^2)]^{1/2}(h/R)\}$, psi
λ = axial wavelength parameter $(m\pi R/L)$
Φ = wavelength parameter $\lambda/S^{1/2}$
γ = specific weight of shell material, lb/in.^{3}
ρ = density of shell material (γ/g, lb-sec^2/in.^{4})
ν = Poisson's ratio of cylinder material
∇^4 = operator $(\partial^4/\partial x^4) + 2(\partial^4/R^2\partial\theta^2\partial x^2) + (\partial^4/R^4\partial\theta^4)$
∇^{-4} = inverse operator defined by $[\nabla^{-4}(\nabla^4 f) = f]$
ω = circular frequency $(2\pi f)$, rad/sec
$\bar{\Omega}$ = frequency parameter $[12(1 - \nu^2)\rho R^4\omega^2/Eh^2]$

Introduction

FOR many years, problems of the free vibrations of unloaded or symmetrically loaded cylindrical shells have interested investigators.[1–7] The problem of the free vibrations of cylindrical shells subjected to asymmetric loads has remained untouched, however. The particular asymmetric loading of a bending moment applied at the boundaries of a cylindrical shell poses an interesting question. Since a compressive load on a cylindrical shell causes a decrease in frequency, and conversely a tensile load causes an increase in frequency, what happens to the frequency spectrum of a cylindrical shell subjected to a bending moment where half the cylinder is under compression and the other half is under tension? The present investigation attempts to answer this question.

In this paper, the vibrations of a cylindrical shell subjected to a bending moment was investigated by linear theory. The effect of hydrostatic pressure acting in conjunction with the bending moment was also considered. Batdorf's variation of the Donnell equation was modified for the free vibration problem and the Galerkin method used to obtain the frequency determinant. An experimental investigation was also performed to check the analytical results.

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Fig. 1 Notation for problem under consideration.

Theory

The Batdorf modification of Donnell's small deflection equation[8] for determining the buckling loads of circular cylinders may be written as

$$Q(w) = D\nabla^4 w + \left(\frac{Eh}{R^2}\right)\nabla^{-4}\left(\frac{\partial^4 w}{\partial x^4}\right) + h\left(\sigma_x \frac{\partial^2 w}{\partial x^2} + \sigma_\theta \frac{1}{R^2}\frac{\partial^2 w}{\partial \theta^2} + 2\tau_{x\theta}\frac{1}{R}\frac{\partial w^2}{\partial w \partial \theta}\right) - q = 0 \quad (1)$$

This equation can be applied to vibration problems of preloaded cylindrical shells if circumferential and longitudinal inertia forces are neglected and the radial surface loading q is replaced by the radial inertia force

$$-\rho h(\partial^2 w/\partial t^2)$$

We shall consider the free vibrations of a cylinder under combined bending, axial load and external lateral pressure for which the stresses in the shell are given by

$$\sigma_x = \sigma_c + \sigma_b \cos\theta$$
$$\tau_{x\theta} = 0 \qquad \sigma_\theta = pR/h \quad (2)$$

where $\sigma_c = P/2\pi Rh$ and $\sigma_b = M/\pi R^2 h$.

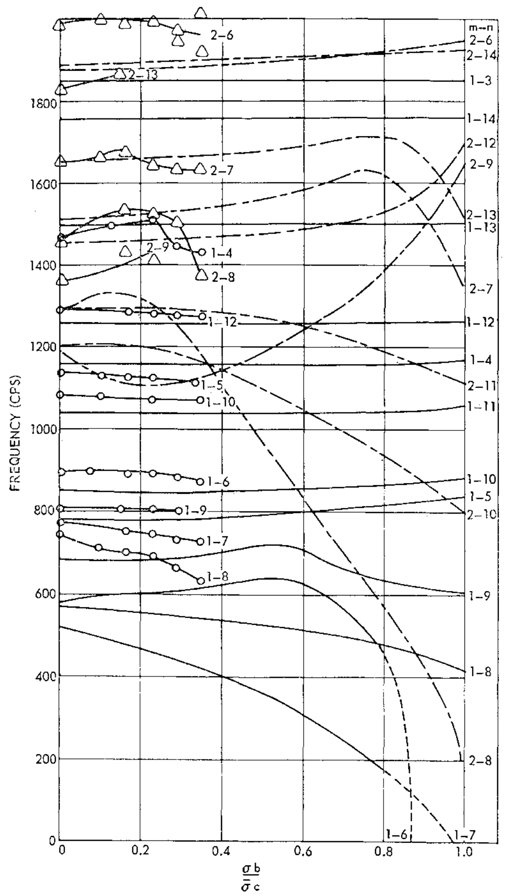


Fig. 2 Variation of frequency with bending moment.

The radial deflection w is assumed to be given by the infinite series

$$w = \sin\frac{m\pi x}{L}\sin\omega t \sum_{n=0}^{\infty} a_n \cos n\theta \tag{3}$$

which, as discussed by Batdorf, satisfy the simple support conditions of the cylinder ends by bulkheads rigid in their own plane but free to warp out of their plane. Using the well-known Galerkin method to satisfy Eq. (1), the following equations for the coefficients a_n are obtained:

$$\int_0^{2\pi}\int_0^{L} Q(w)\sin\frac{r\pi x}{L}\cos s\theta R d\theta dx = 0$$

$$r = 1, 2, 3, \ldots \qquad s = 0, 1, 2, \ldots \tag{4}$$

or, with the aid of Eqs. (1–3),

$$a_n\left\{\left[\Phi + \frac{n^2}{\Phi S}\right]^2 + \frac{1}{[\Phi + (n^2/\Phi S)]^2} + \frac{n^2}{\Phi^2 S}p^* - 2\frac{\sigma_c}{\bar{\sigma}_c} - \frac{\bar{\Omega}}{(\Phi S)^2}\right\} - \frac{\sigma_b}{\bar{\sigma}_c} \tag{5}$$

$$[(1 + \delta_{1n} - \delta_{0n})a_{n-1} + a_{n+1}] = 0 \qquad (n = 0, 1, 2, \ldots)$$

where

$$\delta_{jn} = \begin{cases} 1 \text{ if } n = j \\ 0 \text{ if } n \neq j \end{cases}$$

The frequency spectrum is determined by the condition that the determinant of the coefficients of Eq. (5) vanishes in order for the coefficients a_n to have a nontrivial solution.

Results and Discussion

Computed values of frequency in cycles per second for a particular aluminum cylinder having a radius of 4 in., a radius thickness ratio of 250, and a radius length ratio of 0.524 were obtained for various values of bending moment and a 2-psi external pressure and are shown in Table 1 and in Fig. 2. The material properties were taken as E equal to 10×10^6 psi, γ equal to 0.098 lb/in.^{3}, and $\nu = 0.33$. The frequencies were obtained by truncating the determinant of the coefficients of Eq. (5) and finding the three lowest eigenvalues for each value of m by a matrix iteration technique. It was found by comparing 18×18 and 25×25 determinants for each σ_b/σ_c value that all the tabulated frequencies converged to at least three significant figures. The applied bending moment at which the vibration frequency vanishes

Table 1 Tabulation of analytical frequency spectra (cps) as a function of bending moment parameter ($\sigma_b/\bar{\sigma}_c$)

	Frequency, cps					
$\sigma_b/\bar{\sigma}_c$	0	0.2	0.4	0.6	0.8	1.0
n			$m = 1$			
0	8008	8008	8008	8008	8008	8008
1	5849	5849	5849	5849	5849	5849
2	3233	3233	3233	3233	3233	3233
3	1849	1849	1849	1850	1851	1853
4	1152	1152	1154	1158	1163	1169
5	778	780	789	802	821	841
6	583	601	624	630	466	...[a]
7	520	481	410	317	174	...[a]
8	565	564	542	509	...[a]	413
9	683	686	698	716	623	608
10	846	847	851	858	868	883
11	1040	1040	1042	1045	1050	1056
12	1258	1258	1259	1261	1264	1267
13	1498	1498	1499	1500	1501	1503
14	1759	1759	1759	1760	1761	1762
			$m = 2$			
0	8009	8009	...	8012	8014	8017
1	7332	7332	...	7330	7328	7326
2	5850	5850	...	5851	5851	5851
3	4376	4376	...	4378	4379	4380
4	3237	3237	...	3241	3244	3248
5	2430	2431	...	2441	2449	2459
6	1877	1880	...	1903	1924	1950
7	1511	1519	...	1585	1632	1343
8	1291	1330	...	797	581	190
9	1194	1097	...	1239	1372	1665
10	1203	1204	...	1048	938	805
11	1296	1297	...	...[a]	1183	1114
12	1451	1458	...	...[a]	1524	1755
13	1652	1656	...	1685	1714	1524
14	1889	1891	...	1907	1921	1939
			$m = 3$			
0	8012	8015	...	8042	8062	8087
1	7697	7694	...	7672	7657	7638
2	6884	6884	...	6884	6883	6882
3	5856	5856	...	5858	5859	5861
4	4846	4847	...	4851	4855	4860
5	3973	3974	...	3984	3992	4003
6	3269	3271	...	3291	3308	3330
7	2726	2731	...	2770	2850	2851
8	2326	2337	...	2423	2466	1135
9	2054	2084	...	1250	935	...[a]
10	1897	1936	...	2160	2140	...[a]
11	1845	1699	...	1792	2314	2312
12	1884	1830	...	1551	1363	...[a]
13	1998	2028	...	2463	1677	1540
14	2173	2186	...	1993	...[a]	1849

[a] Values did not converge.

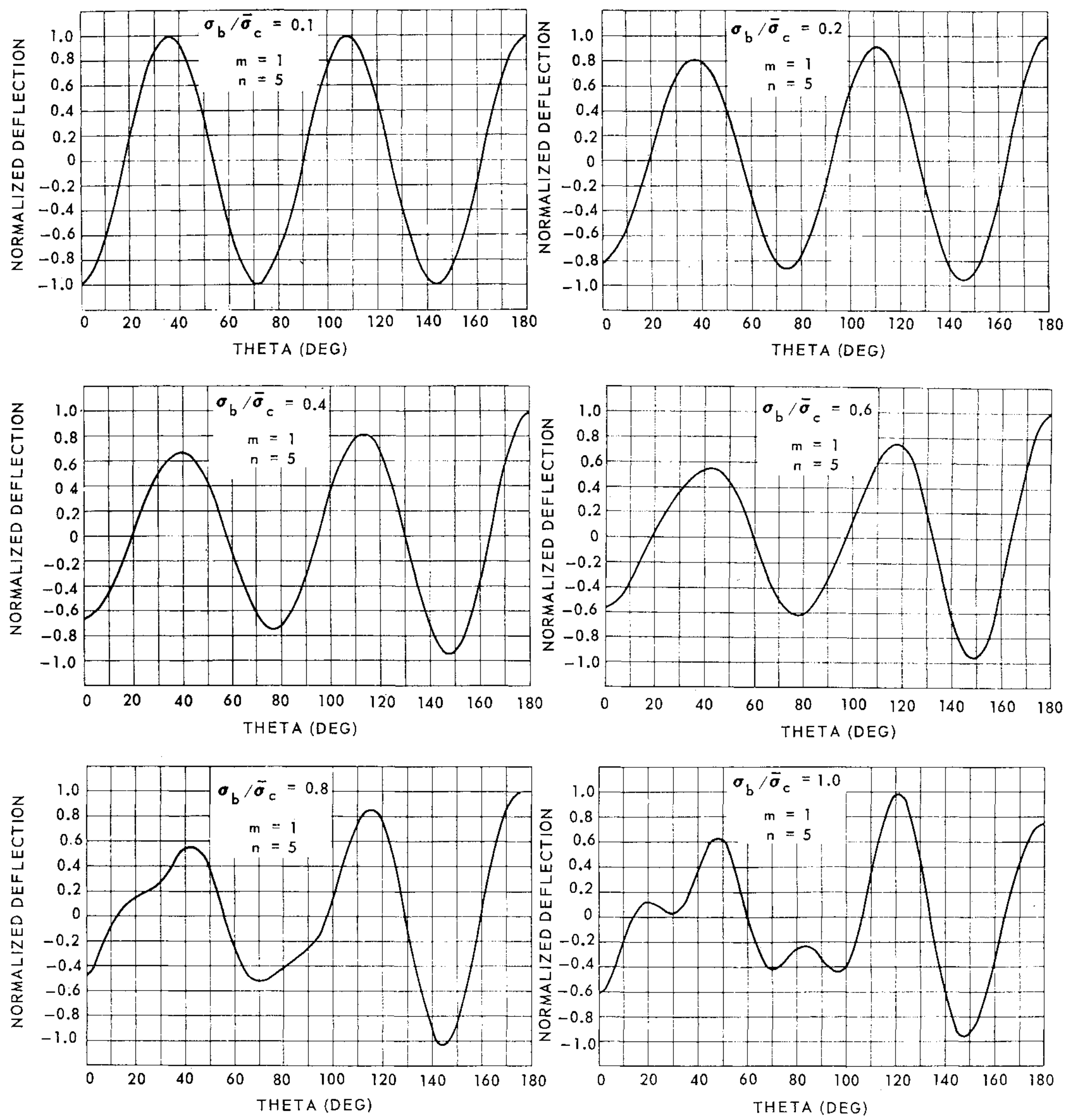


Fig. 3a

Fig. 3 Variation of modal shape with increasing bending moment (Fig. 3b on opposite page).

is the bending moment at which the cylinder buckles and is in good agreement with the results of Ref. 10.

The first problem that arose in the analysis was the identification of modes and frequencies. The mode shapes and frequencies of an unloaded cylinder or a cylinder subjected to uniform axial load or lateral pressure can be identified by an integer corresponding to the number of circumferential waves. For increasing values of σ_b/σ_c, the circumferential modal shape (see Fig. 3) became very irregular, and the number of circumferential waves did not have any real meaning. It was found, however, by taking very closely spaced values of σ_b/σ_c that one could plot smooth frequency curves from σ_b/σ_c equal to zero up to the theoretical buckling value of unity. The value of n for a given curve in Fig. 2 is the value obtained when σ_b/σ_c is equal to zero.

The analytical results plotted in Fig. 2 indicate that as the bending moment becomes larger some of the frequencies increase, whereas others decrease. For example, modes, such as m equal to 1 and n equal to 6–9 show a decrease in frequency with bending moment while modes, such as m equal to 1, n equal to 5, m equal to 2, and n equal to 9 and 12, show an increase in frequency with bending moment. A plot of two modal shapes, one which gives an increase of frequency with bending moment and one which gives a decrease, is shown in Fig. 3. Since the modal shape is symmetrical about the vertical axis of the cross section, only half the deflection shape is shown in Fig. 3. The angle 0° denotes the point of maximum compressive stress, and 90° the point of zero bending stress. Figure 3a, for m equal to 1 and n equal to 5, shows the variation of mode shape when the frequency increases with bending moment. The mode shapes show an increasing maximum deflection on the tension side of the cylinder ($\theta > 90°$) with increasing bending moment. For the case m equal to 1 and n equal to 6 shown in Fig. 3b, the frequency increases until σ_b/σ_c reaches 0.53 (see Fig. 2) and then decreases with increasing moment until buckling occurs. The mode shapes in Fig. 3b indicate increasing maximum deflection initially on the tension side as the frequency increases and then on the compression side as frequency decreases. We may thus suspect that frequency curves that increase with bending moment correspond to maximum deflection on the tension side and curves that decrease with bending moment correspond to maximum deflection on the compression side.

In addition to vibrations symmetric about the top of the shell, antisymmetric vibrations were also investigated, with the radial deflection assumed to be of the form

$$w = \sin\frac{m\pi x}{l}\sin\omega t \sum_{n=1}^{\infty} b_n \sin n\theta \tag{6}$$

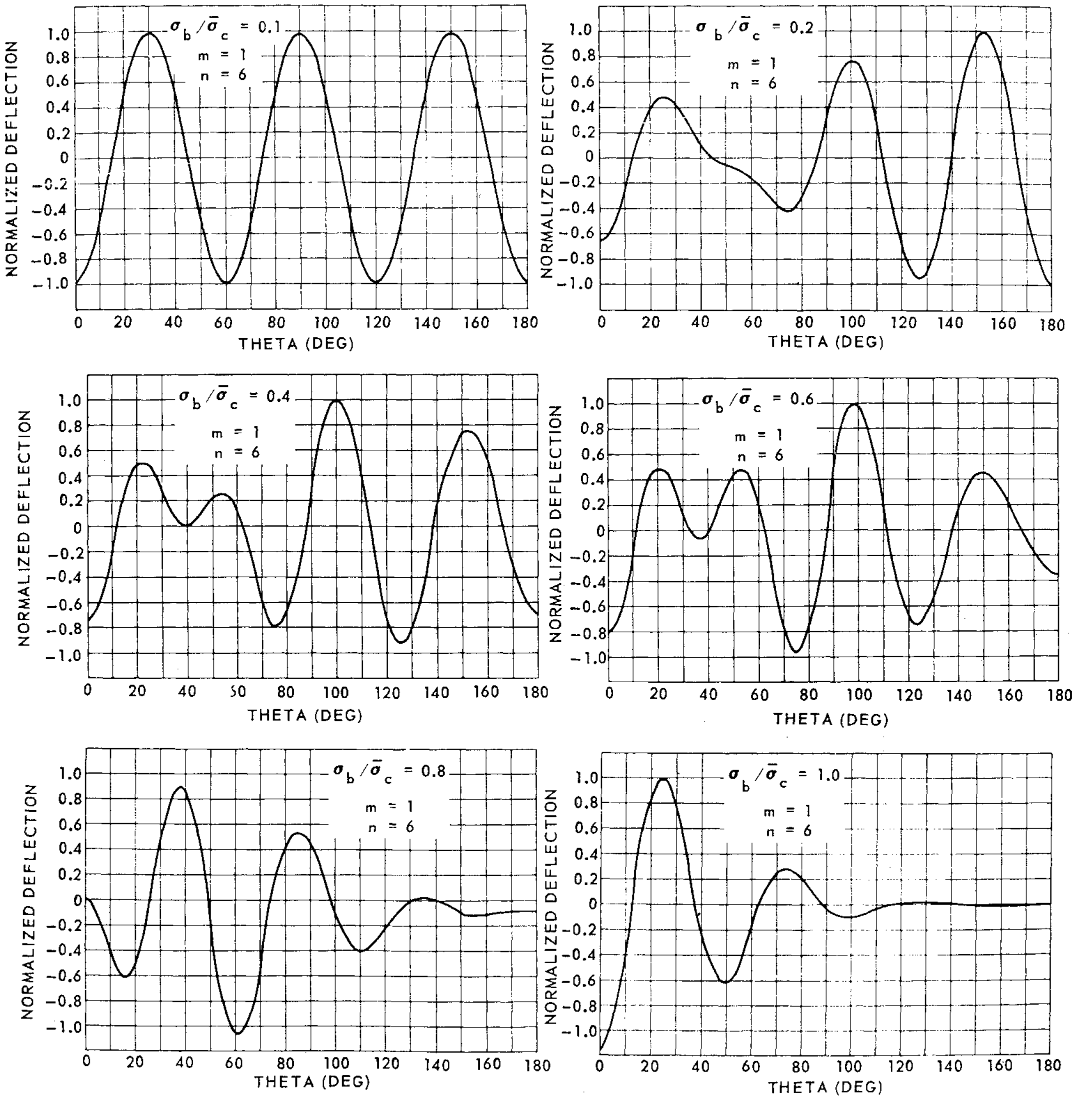


Fig. 3b (Fig. 3a on opposite page.)

The computed frequencies for antisymmetric vibrations were identical with those obtained by using Eq. (3). This result is similar to that obtained by Flügge[9] for the problem of buckling of a cylinder subjected to a bending moment.

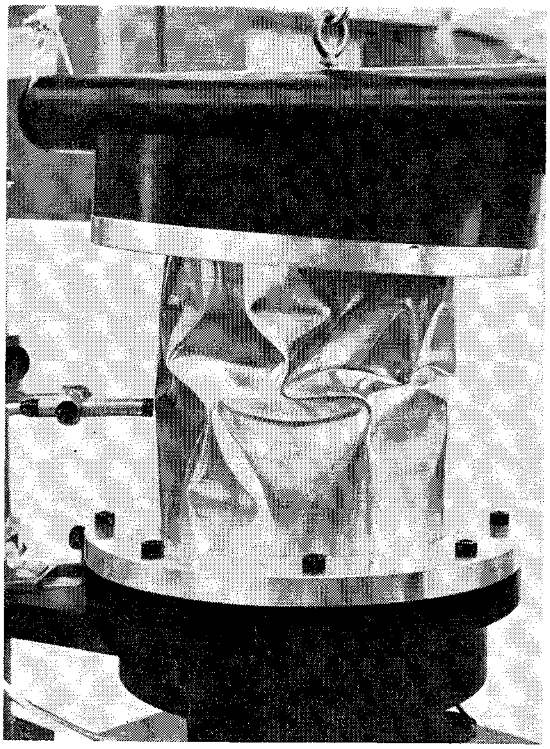
Experimental Investigation

The cylinder used in the experimental investigation had the dimensions and material properties of that of the analytical investigation. It was made of 0.016-in. 6061 aluminum sheet in the T6 condition to insure elastic action and to fit within the load capacity of the test rig. The specimen was first spun on a mandral, its seam was then butt welded, and finally the cylinder was again spun to insure circularity of the cross section. The cylinder was clamped in grooved aluminum end plates filled with cerrobend, a low melting point alloy. External moment was applied to the loading bar by dead weight acting through cables (see Fig. 4a). The loading bar and upper end caps were counterbalanced to eliminate any axial dead weight loading. The loads applied at the ends of the loading bar were equal and opposite so that a pure bending moment was applied to the cylinder. External pressure was applied to the cylindrical shell by using a pump to evaluate the internal air space. The pressure was measured by a monometer.

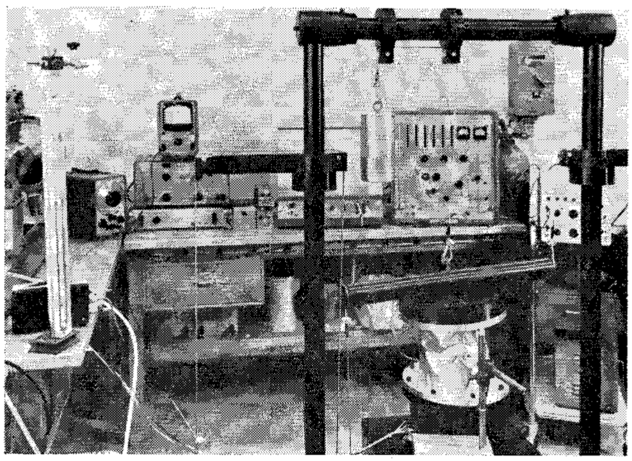
Shell breathing mode vibrations were excited by an electromagnet. A condenser microphone, which has a flat response characteristic from 20 to 10,000 cps, was used to measure the response of the cylinder. The microphone was placed on a track that could be moved along the length of the cylinder and rotated around the cylinder (see Fig. 4b). It was necessary to rotate the microphone since the nodal pattern stayed fixed with respect to the electromagnet.

Table 2 Tabulation of experimental frequency spectra (cps) as a function of a bending moment parameter ($\sigma_b/\bar{\sigma}_c$)

$\sigma_b/\bar{\sigma}_c$	0	0.106	0.168	0.232	0.293	0.355
n			$m = 1$			
4	1471	...		1509	1441	1435
5	1134	1131	1127	1126	...	1112
6	895	...	893	897	883	878
7	782	761	752	748	734	730
8	747	711	703	692	662	634
9	805	...	803	807	800	...
10	1081	1079	...	1076	...	1071
12	1292	1290	1286	1281	1277	1294
N			$m = 2$			
6	1988	2009	1992	1977	1953	1926
					1974	2013
7	1655	1663	1675	1646	1631	1637
8	1459	1447	1541	1525	1511	1374
9	1361	...	1441	1415	...	...
13	1833	...	1868	...	...	...



a)



b)

Fig. 4 Test setup.

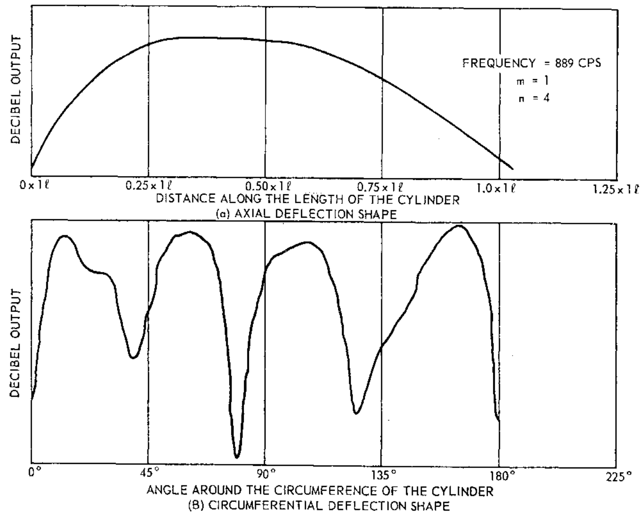


Fig. 5 A typical mode shape record.

The test procedure consisted of first applying the desired external pressure and bending moment to the cylindrical shell. The frequency of the current supplied to the electromagnet was varied by means of an oscillator until a resonant frequency was reached. This frequency was accurately measured by a counter connected in parallel to the microphone. The microphone was then moved first in the axial and then in the circumferential direction around the cylinder. The output of the microphone was rectified and recorded on the y axis of a x-y plotter. The position of the microphone was measured by means of a balance circuit with an output voltage proportional to its movement and was recorded on the x axis of the x-y plotter. A typical record appears in Fig. 5. The minimum points appearing in this figure are node points, since the microphone voltage output is approximately proportional to radial displacement. The applied moment was increased in steps and the frequency spectrum obtained until buckling occurred. The test cylinder in the buckled state is shown in Fig. 4.

The experimental results for the frequency spectrum of the test cylinder under a bending moment and a 2-psi external pressure appear in Table 2. These results are also plotted in Fig. 2. The experimental results in all cases fall above the analytical curves. This is to be expected since the analysis was done for cylinders with simply supported ends, whereas in the experiment, the specimen ends are somewhere between clamped and simply supported. The experimental results, however, qualitatively check the analysis. For m equal to 1, n equal to 6–9, and m equal to 2, n equal to 8, the experimental data show a decrease of frequency with bending moment, whereas for m equal to 2, n equal to 9, the data indicate an increase of frequency with bending moment as predicted by the analysis.

No experimental results were obtained beyond a $\sigma_b/\bar{\sigma}_c$ equal to 0.355, since buckling occurred at a value of $\sigma_b/\bar{\sigma}_c$ equal to 0.43. This prevented a check of the large increases and decreases at frequencies occurring beyond this point. A logical extension of this work would be to investigate the combined effect of internal pressure and bending moment, since the buckling value of a test cylinder under this loading would be very close to that predicted by small deflection theory (see Ref. 10), and experimental frequencies could then be obtained up to this point.

References

[1] Arnold, R. N. and Warburton, G. B., "Flexural vibrations of the walls of thin cylindrical shells having freely supported ends," Proc. Roy. Soc. London **A197**, 238–256 (1949).

[2] Arnold, R. N. and Warburton, G. B., "The flexural vibrations of thin cylinders," Proc. Inst. Mech. Engrs. **167**, 62–74 (1953).

[3] Weingarten, V. I., "Experimental investigation of the free vibrations of multi-layed cylindrical shells," Exptl. Mech. **4**, 200–206 (July 1964).

[4] Weingarten, V. I., "On the free vibration of thin cylindrical shells," AIAA J. **2**, 717–722 (1964).

[5] Federhofer, K., "Uber die Eigenschwingungen der Axial gedrückten kreiszylinderschale," Sitzungsber, Akad. Wiss. Wien, **145** (1936).

[6] Fung, Y. C., Sechler, E. E., and Kaplan, A., "On the vibration of thin cylindrical shells under internal pressure," J. Aeronaut. Sci. **24**, 650–660 (1957).

[7] Koval, L. R. and Cranch, E. T., "On the free vibrations of a thin-walled circular cylindrical shell subjected to an initial static torque," *Proceedings of the Fourth U.S. National Congress of Applied Mechanics* (American Society of Mechanical Engineers, New York, 1962), pp. 107–117.

[8] Batdorf, S. B., "A simplified method of elastic-stability analysis for thin cylindrical shells, II—Modified equilibrium equation," NACA TN 1342 (June 1947).

[9] Flügge, W., "Die Stabilitat der Kreiszylinderschale," Ing.-Arch. **3**, 463–506 (1932).

[10] Weingarten, V. I., "Effects of internal pressure on the buckling of circular-cylindrical shells under bending," J. Aerospace Sci. **29**, 804–808 (1962).